# Influence of magnetic cohesion on the stability of granular slopes

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We use a molecular dynamics model to simulate the formation and evolution of a granular pile in two dimensions in order to gain a better understanding of the role of magnetic interactions in avalanche dynamics. We find that the angle of repose increases only slowly with magnetic field; the increase in angle is small even for intergrain cohesive forces many times stronger than gravity. The magnetic forces within the bulk of the pile partially cancel as a result of the anisotropic nature of the dipole-dipole interaction between grains. However, we show that this cancellation effect is not sufficiently strong to explain the discrepancy between the angle of repose in wet systems and magnetically cohesive systems. In our simulations we observe shearing deep within the pile, and we argue that it is this motion that prevents the angle of repose from increasing dramatically. We also investigate different implementations of friction with the front and back walls of the container, and conclude that the nature of the friction dramatically affects the influence of magnetic cohesion on the angle of repose.

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## I. INTRODUCTION

There is much current interest in the dynamics of granular systems [1]. Recent research has focused on understanding the stability of granular piles and the characterization of dense granular flows [2].

An important measure of pile stability is the angle of repose. If grains are poured onto a flat surface, they form a cone with a well-defined angle, characteristic of the material. If a granular pile is tilted or if more material is added to the top, the angle increases until the pile becomes unstable. An avalanche then occurs, and the system relaxes through rearrangement of grains within the pile. If the rate of adding grains is increased, fully developed granular flows result. Both the slope angle and the avalanche dynamics are known to depend upon the dimensionality of the pile [3] and the confining geometry [4].

The majority of investigations into granular flows have focused on dry granular media in which the interaction between grains is only through collisions. In many systems cohesion exists due to the presence of an interstitial liquid [5], the influence of magnetic or electrostatic interactions [6], or, for fine powders, the effects of van der Waals forces [7]. Cohesion in granular piles can influence the slope angle [8], the packing fraction [9], and the flow dynamics [10]. In particular, magnetic cohesion can be used as a control parameter to modify macroscopic properties of the pile.

Another field in which cohesion is important is in the separation of binary mixtures. Recent research in magnetic separation techniques has enabled the vibration-driven separation of binary mixtures to be enhanced by using strong inhomogeneous magnetic fields [11]. However, a detailed understanding of the effects of magnetic cohesion is still lacking.

In this paper we describe molecular dynamics simulations carried out to investigate the influence of magnetic cohesion on the stability of granular slopes. In Sec. II we briefly discuss the angle of repose as a measure of cohesion and review recent literature. Section III provides the details of our simulation, and in Sec. IV we present and discuss our results. We find that the weak dependence of the repose angle on magnetic cohesion can be understood in terms of the dynamics of the pile and the influence of friction with the front and back walls of the container.

## II. ANGLE OF REPOSE AS A MEASURE OF COHESION

A generally used measure of the effect of cohesion is the angle of repose of a granular pile. As the particles become more cohesive, the angle of a granular slope might be expected to increase. It is possible to define a cohesion strength $R$ [12,13] as the ratio of the maximum cohesive force $F_v$ between two particles in contact and the particle weight. In the absence of cohesion, the angle of repose is independent of the weight of the particles as frictional forces scale linearly with the particles' weight. In this case $R=0$. When $R>1$, the cohesive force is greater than the particles' weight and one particle can be suspended from another. As $R$ increases from zero, the slope of a pile will be increasingly affected by cohesion, and one might expect the slope angle to approach 90° for $R>1$.

Forsyth *et al.* [12] have carried out a series of experiments investigating the influence of magnetic cohesion on repose angle. They poured steel ball bearings into a narrow box to measure the angle of repose, $\theta_r$, in a uniform vertical magnetic field. They found that $\theta_r$ increased slowly and linearly with the magnetic field strength. The increase in $\theta_r$ was only a few degrees even when the interparticle cohesive forces were many times greater than the particles' weight.

Fazekas *et al.* [13] used a two-dimensional molecular dynamics model to simulate the experiments of Forsyth *et al.*, treating the particles as point dipoles aligned with a uniform vertical magnetic field. The results showed a slow increase in the angle of repose with magnetic field strength, at a rate of 0.5° per unit $R$. Even though the experiments were in three dimensions and the simulations were in two dimensions, there was good quantitative agreement in the rate of increase of $\theta_r$. However, the value of $\theta_r$ in the absence of a magnetic field was substantially lower (19° rather than 31°) in the simulations. This discrepancy was attributed to the effects of friction between the particles and the front and back walls of the container.

The angle of repose of dry spheres is generally measured as about 23° (see [5] and references therein). The value of 31° obtained by Forsyth *et al.* is rather high, and this can be attributed to the narrowness of the container (width of 5 particle diameters). Forsyth *et al.* found that the repose angle decreased when they used a wider container. A detailed experimental investigation of the influence of sidewalls on the repose angle has been carried out by Nowak *et al.* [4].

In contrast with magnetic systems, experiments on wet granular materials show a dramatic increase in the angle of repose when a small quantity of liquid is added [5,8,14]. Liquid bridges have been observed to form between particles in contact, providing a cohesive force. It is, however, difficult to directly relate the quantity of liquid to the interparticle force. Albert and co-workers [5,8] measured the angle of repose of spherical glass particles with varying amounts of oil added. They fitted their data using a model based on the stability of a particle on the surface of a pile, treating the volume of the liquid bridges as an unknown parameter. They found that the slope angle approached 90° at $R=1$ and the rate of increase $d\theta_r/dR$ was 58° per unit $R$.

The increase in $\theta_r$ in magnetic systems is a very small effect; $d\theta_r/dR$ is two orders of magnitude smaller than in wet granular systems. One would intuitively expect magnetic cohesion to have a more dramatic effect on the system, as occurs with liquid-bridge cohesion, but this appears not to be the case. To our knowledge, nowhere in the literature has anyone offered a satisfactory explanation of this discrepancy.

## III. DETAILS OF THE SIMULATIONS

Our two-dimensional molecular dynamics model follows the scheme of Cundall and Strack [15]. The particles were modeled as spheres with an approximately Gaussian distribution of diameters, with a mean value of $d=0.8$ mm and a standard deviation of $\sigma=0.03$ mm. The distribution was curtailed at $3.35\sigma$, so that all diameters lie in the range 0.7–0.9 mm. The induced magnetic dipole moments were always aligned with the external field, though the particles themselves could rotate in the plane of the container. The simulation parameters are listed in Table I.

We used a Hertzian contact model, with a nonlinear damping force in the normal direction to model the dissipation of energy in collisions. We varied the effective coefficient of restitution $e$ between 0.6 and 0.95, and found that the value of $e$ had no significant effect on our results. The friction force in the tangential direction was set to the minimum of $\mu F_n$ and $\lambda v_t$, where $\mu$ is the Coulomb friction coefficient, $F_n$ is the normal contact force, $\lambda$ is a viscous friction coefficient, and $v_t$ is the relative tangential velocity of the point of contact. The values of $\mu$ and $\lambda$ are given in Table I. The time step was chosen to be 5 $\mu$s, significantly shorter than the typical duration of a collision. We also compared results using different contact and friction models: a Hertzian contact model with stick-slip friction and a linear spring model with Coulomb friction. The choice of contact and friction models had no significant effect on the simulation results.

TABLE I. Parameters used in the simulation.

| Symbol | Parameter | Value | Unit |
|---|---|---|---|
| $d$ | Mean particle diameter | 0.8 | mm |
| $L$ | Container width | 40 | mm |
| $k$ | Spring constant in linear model | 9425 | N m$^{-1}$ |
| $E$ | Young's modulus | 0.015 | GPa |
| $\rho$ | Density | 7500 | kg m$^{-3}$ |
| $g$ | Acceleration due to gravity | 9.81 | ms$^{-1}$ |
| $e$ | Coefficient of restitution | 0.95 | |
| $\mu$ | Particle-particle coefficient of friction | 0.5 | |
| $\mu_w$ | Particle-wall coefficient of friction | 0.5 | |
| $\lambda$ | Viscous friction coefficient | 10 | kg s$^{-1}$ |
| $\Delta t$ | Integration time step | 5 | $\mu$s |

To simulate the formation of a granular pile, particles were introduced into the system, one every 3000 time steps. Each new particle was released with zero velocity on the left side of a container of width of 50 particle diameters, at a height just greater than that of the highest existing particle in the pile. Hence newly introduced particles had a low momentum and did not significantly disturb the pile upon impact. Particles colliding with the base of the container became stuck, forming an uneven surface upon which the pile was constructed. Particles reaching the right side were removed from the system. Figure 1 shows a snapshot of the simulation.

We used different methods to model the front and back walls of the container in our simulations. First, we compared our results to those of Fazekas *et al.* [13] in the absence of front and back container walls.

Second, we directed a small percentage $p$ of each normal contact force $F_n$ on each particle outward, as if the particles were exerting a force $pF_n$ on the front and back walls. Particles experienced friction $\mu_w pF_n$, where the particle-wall friction coefficient $\mu_w$ was set to 0.5. The percentage of force directed outward was a parameter of the simulations.

Third, we simulated the effect of front and back walls by treating the particles as sliding against the walls. The par-

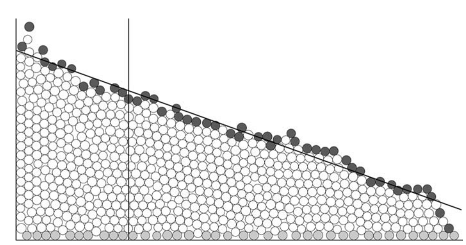

FIG. 1. A snapshot of a granular slope in the absence of a magnetic field. The diagonal line is a fit to the surface particles (darkly shaded). The lightly shaded particles adhere to the base of the container. The vertical line, a quarter of the container width from the left wall, is the position at which we evaluate the particles' velocities and magnetic forces as a function of depth in the pile, as discussed in Sec. IV.

ticles experienced a constant drag force $\alpha mg$, which was proportional to the particle weight and opposed the direction of motion. Rotational drag was neglected. We treated the drag constant $\alpha$ as a variable parameter.

To determine the angle of repose of the pile, the width of the container was divided into bins and the highest particle in each bin identified. A least-squares straight-line fit was applied to these particles (see Fig. 1).

The particles in our simulations were spherical and weakly magnetic, with moments induced by and parallel to a uniform vertical magnetic field $\mathbf{B}$. It is well known that the magnetic field due to a homogeneous sphere with total magnetic moment $\mathbf{m}$ in a uniform field is equal to that of a point dipole with the same magnetic moment located at the sphere's center [16]. For weakly magnetic particles, the susceptibility $\chi$ is low ($\chi \ll 1$), such that the magnetic moment induced in each particle is too small to affect the uniformity of the field experienced by other particles. We therefore treat our spheres as a collection of interacting point dipoles.

The interaction energy $E$ between two point dipoles of magnetic moment $\mathbf{m}$ separated by $\mathbf{r}$ is

$$E = \frac{\mu_0 |\mathbf{m}|^2}{4\pi |\mathbf{r}|^3}(1 - 3\cos^2\phi), \quad (1)$$

where $\phi$ is the angle between the direction of the magnetic field and the vector $\mathbf{r}$, and $\mu_0$ is the permeability of free space [16]. The magnetic dipole-dipole force between two spheres has been measured [17] and found to be in good agreement with Eq. (1). The magnetic force is highly anisotropic; its sign changes depending on the relative position of the particles in the magnetic field. It is also relatively short range, decaying as $1/|\mathbf{r}|^4$. In our simulations we use a cutoff of 6.25 diameters, beyond which we consider the magnetic forces to be negligible [18]. At this distance the magnetic forces are over three orders of magnitude smaller than for particles in contact.

Consider the interaction between two equal spheres in contact, with diameter $d$, volume $V$, and magnetic dipole moment $\mathbf{m} = \chi V \mathbf{B}/\mu_0$. When $\mathbf{r}$ is parallel to $\mathbf{B}$, the particles attract with a maximum cohesive force of magnitude $F_v = \pi\chi^2 B^2 d^2/24\mu_0$. When $\mathbf{r}$ is perpendicular to the magnetic field, the particles repel with a force of half the magnitude, $F_h = F_v/2$. The cohesion strength $R$, defined as the ratio of the maximum cohesive force $F_v$ between two particles in contact and the particle weight, is given by

$$R = \frac{F_v}{mg} = \frac{\chi^2 B^2}{4\mu_0 \rho d g}, \quad (2)$$

where $\rho$ is the density of the particles and $g$ the acceleration due to gravity.

## IV. RESULTS AND DISCUSSION

### A. Validation of the model

First, we validate our model by repeating the simulations of Fazekas *et al.* [13] in a system with no front and back walls. We ran the simulation for 180 s (simulated time), during which 12 000 particles were introduced into the system.

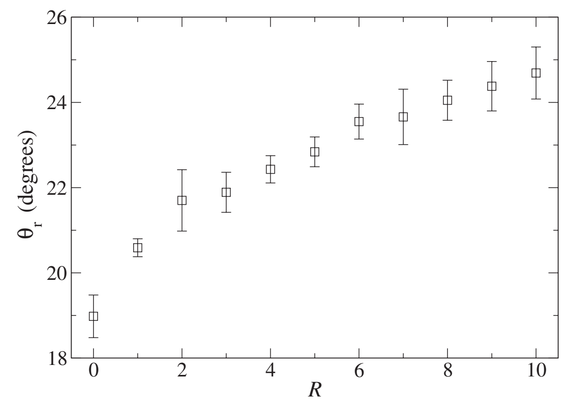


FIG. 2. Angle of repose as a function of cohesion strength $R$ in a vertical magnetic field in a container of width of 50 particle diameters.

We used a range of values of the cohesion strength $R$ between 0 and 10. We consider that at higher field strengths than this, the slope is insufficiently smooth for a single angle to be a suitable parameter to describe the system. Figure 2 shows the angle of repose as a function of the cohesion strength $R$. The angle increases slowly with cohesion, with an increase of only a few degrees even when the cohesive forces are ten times as great as the particles' weight. Our simulations show a linear dependence of the angle of repose on $R$, but with a nonlinearity below $R=2$. Our simulations yielded a value of $d\theta_r/dR = 0.50°$, which is in good agreement with simulations of Fazekas *et al.* [13].

The effect of cohesion on $\theta_r$ is weak: when $R=1$, magnetic and gravitational forces are of the same magnitude, and one might expect the angle of repose to be substantially greater than in the case of zero cohesion. It has been suggested that the weak dependence of $\theta_r$ on $R$ is a result of the anisotropic nature of the cohesive force [12,13]. We have discovered that, due to this anisotropy, the field-induced magnetic dipole-dipole forces in the bulk of the pile partially cancel each other out. In the following section, we investigate this cancellation effect to determine whether it provides a sufficient explanation for the weak dependence of the repose angle on magnetic cohesion in our simulations.

### B. Magnetic cancellation

The cohesion strength $R$ overestimates the forces in the system. Because of the anisotropic nature of the magnetic dipole-dipole force, the average force between two particles in contact is less than the maximum force $F_v$. The force changes sign depending on the angle $\phi$ between $\mathbf{B}$ and $\mathbf{r}$, so the forces acting on a particle due to its surrounding particles can be either attractive or repulsive.

In three dimensions, magnetic forces can cancel exactly. We have calculated the net magnetic force on a point dipole above an infinite plane of magnetic material, in a uniform vertical magnetic field. The attraction of the dipole to material underneath is counteracted by repulsion from material to the sides. The forces cancel exactly, and the dipole experiences no net force.

The analogous calculation in two dimensions (the net magnetic force on a point dipole due to an infinitely long line) demonstrates partial cancellation. The net force is nonzero, but significantly less than $F_v$.

A dipole above an infinite layer of point dipoles arranged in a regular lattice also experiences partial cancellation in both two and three dimensions. The magnetic dipole-dipole force is a short-range force, decaying as $1/r^4$. Hence the force on a particle will depend very sensitively on the arrangement of its neighboring particles, but only weakly on the arrangement of particles farther away.

As a measure of the magnetic force on a particle in the bulk of a pile, we calculate the sum of the radial components of the magnetic forces due to all other particles. By differentiating the interaction energy, Eq. (1), we obtain the radial component $\mathbf{F_r}$ of the magnetic dipole-dipole force between two particles:

$$\mathbf{F_r} = -\frac{\partial E}{\partial r}\hat{\mathbf{r}} = F_v \frac{d^4}{2r^4}(1 - 3\cos^2\phi)\hat{\mathbf{r}}. \quad (3)$$

We use an algebraic sum of the magnitudes of the radial forces to give an estimate of the cohesion in the packing. A vectorial sum would be close to zero, but the cohesive forces are always present and act to oppose the particles' motion.

We calculate the total radial magnetic force $F_{\text{total}} = \Sigma|\mathbf{F_r}|$ on a particle due to its six nearest neighbors, assuming regular hexagonal packing. By summing the contributions from all six neighbors, we obtain the net cohesive force $F_{\text{total}} = 1.5F_v$. This is true for any orientation of the hexagon relative to the magnetic field direction. Now we add the contributions to the total radial force from the next-nearest neighbors. Consider another ring of particles added around the outside of our original hexagon. The total radial force including contributions from next-nearest neighbors is $F_{\text{total}} = 1.76F_v$.

We have also calculated the total radial magnetic force per particle in our simulations and plotted the results in Fig. 3 as a function of vertical position in the pile. The simulation results show that $F_{\text{total}}$ is approximately constant in the bulk of the pile and agrees well with our calculated value of $1.76F_v$. Note that if the magnetic forces were always attractive, the corresponding net force would be $7.04F_v$. This suggests that the cohesion strength $R$ overestimates the magnetic forces in the system by a factor of 4.

Forsyth *et al.* [12] and Fazekas *et al.* [13] suggest that magnetic anisotropy could be an explanation for the two-orders-of-magnitude discrepancy between the size of the effects of cohesion on magnetic systems and wet systems. Our calculations suggest that this is not the case, and in the next section we outline an alternative explanation for the weakness of the effect of magnetic cohesion.

### C. Avalanche dynamics

In steady fully developed flows in three-dimensional (3D) piles, most of the motion occurs in a surface layer with a linear velocity profile, and there is creep motion in the bulk that decays exponentially [19]. Aguirre *et al.* [3] report that in 2D experiments in a slowly tilted bed, the velocity profile is either purely exponential or a product of an exponential and a Gaussian. Another system that exhibits a predominantly exponential velocity profile is a collection of monodisperse spheres in a slowly sheared 3D Couette cell [20].

In the absence of a magnetic field, we have also observed shearing deep within the pile. In a magnetic field the surface of the heap is more rugged and the size of surface irregularities increases with cohesion. Clusters of regularly packed particles form and move as a block, both on the surface and in the bulk. Shear in the bulk occurs at the boundaries between clusters. The size of the clusters increases with cohesion, and contacts between neighboring particles last for longer than in the absence of magnetic cohesion.

Figure 4 plots the mean particle velocity in our simulations as a function of depth in the pile. In the absence of a

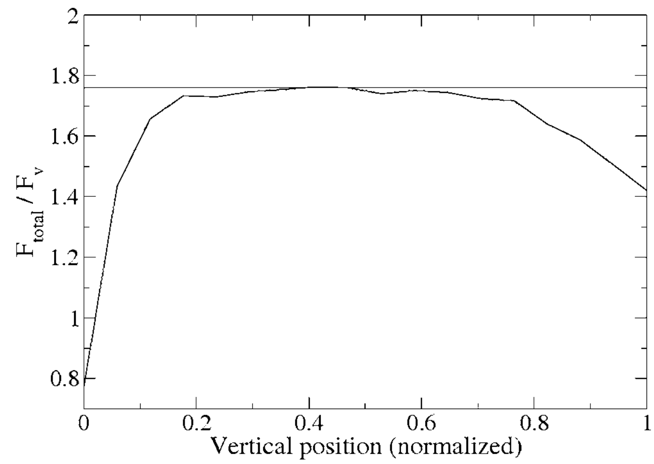


FIG. 3. Total radial magnetic force $F_{\text{total}}$ per particle as a function of vertical position in the pile, measured at a horizontal position a quarter of the container width away from the left wall (the vertical line in Fig. 1). The horizontal line on the graph is $1.76F_v$. The particle positions are normalized so that the bottom of the pile is 0 and the top is 1.

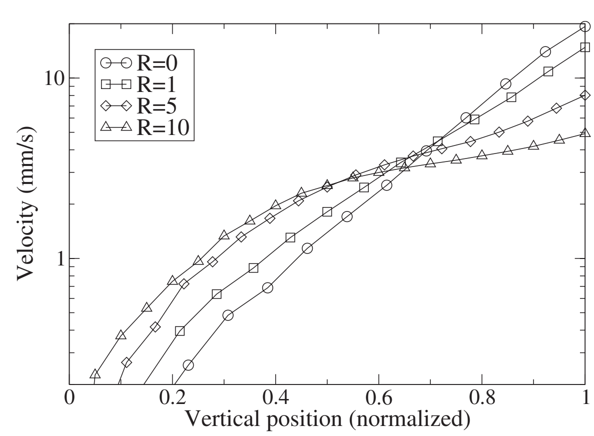


FIG. 4. The mean velocity per particle as a function of vertical position in the pile, measured at a horizontal position a quarter of the container width away from the left wall (the vertical line in Fig. 1). The particle positions are normalized so that the bottom of the pile is 0 and the top is 1. The addition of a magnetic field shifts the motion farther down into the bulk of the pile.

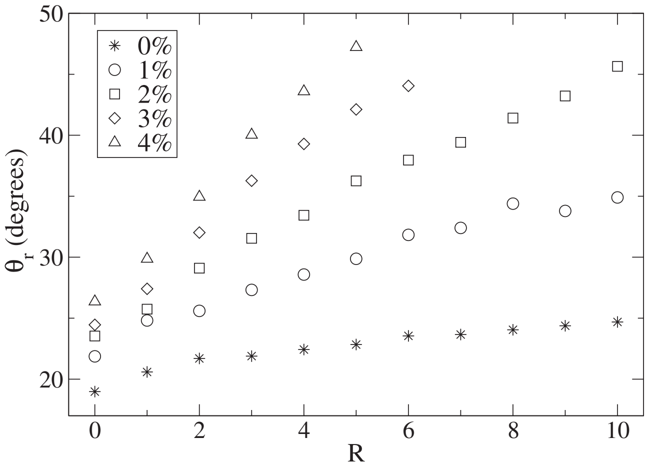


FIG. 5. Angle of repose as a function of cohesion strength $R$ in a vertical magnetic field in a container of width of 50 particle diameters. Results are shown for different values of the friction percentage $p$.

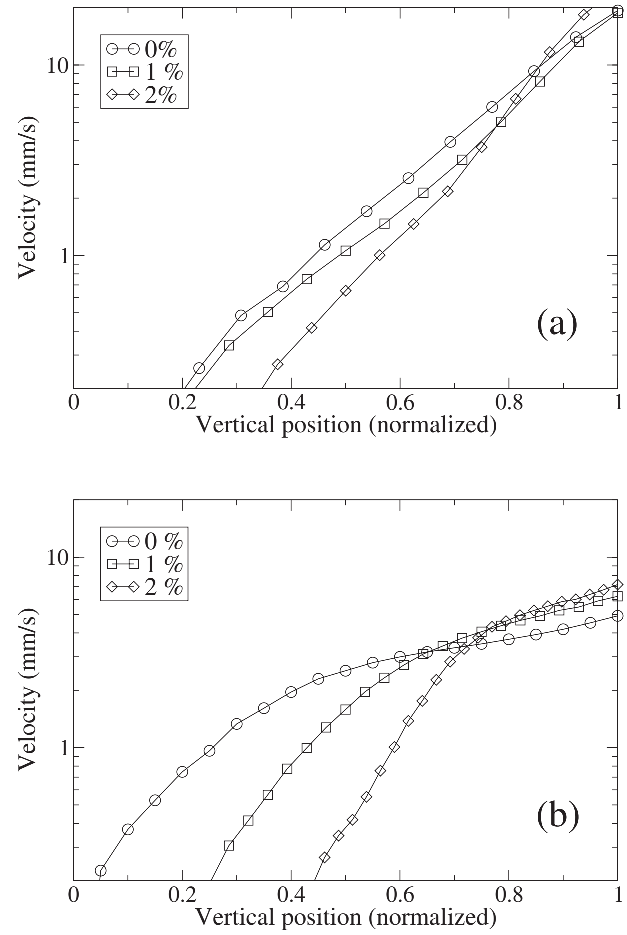


FIG. 6. The mean velocity per particle as a function of vertical position in the pile, measured at a horizontal position a quarter of the container width away from the left wall (the vertical line in Fig. 1). The particle positions are normalized so that the bottom of the pile is 0 and the top is 1. Results are shown for (a) $R=0$ and (b) $R=10$ at different values of the friction percentage $p$.

magnetic field, the velocity decays exponentially with depth. The rate at which we add particles is slow enough that there is no constantly moving surface layer, and the zero-field velocity profile is approximately exponential. On average, the tendency to slip at any given depth is proportional to the weight of particles above that depth. The frictional force that opposes slip, however, is also proportional to the weight of particles above. Hence the weight cancels out of the force balance equation and slip can occur at any depth in the pile.

In the presence of a magnetic field the motion shifts further down into the pile and the shape of the velocity profile changes, as can be seen in Fig. 4. This is because the interparticle cohesive forces in the bulk of the pile do not depend upon depth. Near the surface of the pile, cohesive forces can readily support the weight of the particles above, resulting in less shear than in the absence of cohesion. Farther down in the pile the cohesive forces are less able to support the weight of the particles above, resulting in increased shear. It is this shear deep within the pile that prevents the angle of repose from increasing dramatically.

Restagno *et al.* [21] report a study of the dependence of cohesion on the normal force and how this affects the failure of a granular pile. Their continuum analysis predicts that when the cohesive force is independent of the normal contact force between particles, the pile will fail at depth. The velocity profiles obtained from our simulations confirm this prediction. Restagno *et al.* also argue that when the cohesive force varies linearly with normal contact force, as in the case of liquid bridges, the pile is predicted to fail at the surface. This happens in the "granular regime" in the experiments of Tegzes *et al.* [14] with small amounts of liquid added to the grains. The dependence of cohesion on normal contact force could explain the discrepancy between $d\theta_r/dR$ in our simulations and in liquid-bridge experiments, because, as noted above, it is shear deep in the pile that prevents the angle of repose from increasing substantially.

### D. Effect of friction with front and back walls

So far, we have considered an idealized 2D system. Any real physical system will be influenced by the container walls. It is well known that friction with confining walls can influence both the angle of repose and the velocity profile of avalanches in a narrow box [22,23].

We introduce wall effects into our 2D simulations by using two different friction models, as described in Sec. III. In the first model, a percentage $p$ of each normal contact force is directed toward the front and back walls. The frictional force depends upon the depth within the pile because a particle supports the weight of other particles resting on it. In the second model, a constant drag force proportional to the particles' weight is applied in the opposite direction to the particles' motion. In contrast with the first model, friction is independent of a particles' position in the pile.

Figure 5 shows the angle of repose, $\theta_r$, as a function of cohesion strength $R$ for a range of values of $p$. The repose angle at zero field increases dramatically with $p$ because the depth-dependent friction increasingly opposes motion farther down in the pile. The variation of $\theta_r$ with $R$ is much greater for higher values of $p$. In fact, the gradient $d\theta_r/dR$ increases linearly with $p$.

This behavior can be understood by considering where the motion occurs in the pile. Figure 6 plots the mean particle

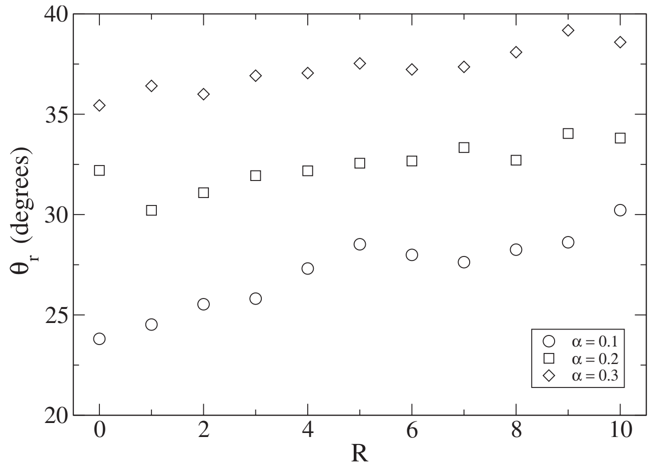


FIG. 7. Angle of repose as a function of cohesion strength $R$ in a vertical magnetic field in a container of width of 50 particle diameters. Particles slide against the front and back walls of the container and are subject to a drag force $\alpha mg$ proportional to the particle weight. Results are shown for different values of the drag constant $\alpha$.

velocity as a function of depth. In the absence of front and back walls ($p=0$), increasing the magnetic field shifts the motion further down into the pile, as explained in Sec. IV C. However, if $p$ is nonzero, the frictional forces with the walls oppose motion in the bulk, causing the velocity profile to change shape and the motion to shift closer to the surface of the pile. Hence there is less motion in the bulk and $\theta_r$ increases more quickly than in the absence of depth-dependent friction.

The effect of constant drag friction, however, is quite different. Figure 7 shows the repose angle $\theta_r$ as a function of $R$ for different values of the drag constant $\alpha$. The angle in zero field is about 24° for $\alpha=0.1$, significantly higher than in the absence of front and back wall friction, and increases further for higher values of $\alpha$. The repose angle $\theta_r$ increases by only a small amount with cohesion. In fact, the gradient $d\theta_r/dR$ $=0.5°$, the same as in the case with no front and back walls. Observing the simulations running, we can see that motion happens deep in the pile, not just near the surface.

The particle velocity profile as a function of depth in the pile is very similar to Fig. 4, demonstrating that this implementation of friction with the front and back walls does not change where slip occurs. Shear is still happening deep within the pile, preventing the angle of repose from increasing dramatically with magnetic cohesion.

### E. Application to three-dimensional systems

The shear deep in the bulk of the pile in the absence of front and back walls explains why the dependence of repose angle on magnetic cohesion is weak. The presence of depth-dependent wall friction, however, results in a much larger gradient $d\theta_r/dR$. The inclusion of wall friction is an attempt to model the three-dimensional nature of many experimental geometries.

There is good agreement between $d\theta_r/dR$ in idealized two-dimensional simulations (both our own and those of Fazekas *et al.* [13]) and in the experiments of Forsyth *et al.* [12], but when wall friction effects are included in the simulations, there is no longer any agreement. One possible explanation is that the simulations were carried out using weakly magnetic particles, for which the point dipole approximation is valid, whereas Forsyth *et al.* used iron spheres, which are ferromagnetic and have a susceptibility $\chi \gg 1$.

Another possible explanation for the apparent discrepancy is that the simulations do not account for magnetic interactions in the third dimension, perpendicular to the front and back walls. We have estimated the magnetic force on a particle close to the front or back wall of a 3D container, due to other particles in the container, assuming that all particles are weakly magnetic. The horizontal component of the magnetic dipole-dipole force between two point dipoles of moment $\mathbf{m}$ and separated by $\mathbf{r}$ is given by

$$\mathbf{F_x} = -\frac{\partial E}{\partial x}\hat{\mathbf{x}} = \frac{3\mu_0|\mathbf{m}|^2}{4\pi|\mathbf{r}|^4}\sin\phi(1-5\cos^2\phi)\hat{\mathbf{x}}, \qquad (4)$$

where $\hat{\mathbf{x}}$ is a unit vector in the direction perpendicular to the wall. We calculate the contributions to the horizontal force on a point dipole from all volume elements in the bulk and integrate over the infinite half-space with $x>0$ and $|\mathbf{r}|$ $>d/2$. (The vertical component to the magnetic force cancels out due to symmetry.) We find that the dipole experiences an attractive force into the bulk of $0.147F_v$.

Thus, particles close to the front and back walls are attracted toward the bulk and away from the walls. We speculate that this attraction will reduce the effect of wall friction and that the system will behave more like our idealized 2D simulations. This may be the cause of the weakness of the effect of magnetic cohesion on the angle of repose observed experimentally.

Recent experiments on granular avalanches in confined geometries subject to electric fields demonstrate that electric cohesion and wall interactions can significantly influence the repose angle [24]. It would therefore be interesting to investigate whether magnetic systems exhibit similar behavior, as suggested by our calculations in this section.

## V. CONCLUSION

We have used a 2D molecular dynamics simulation to investigate the effect of magnetic cohesion on the repose angle of a granular pile. We found that the repose angle increases linearly with cohesion strength $R$. The effect is weak, even when magnetic forces are 10 times as strong as gravity.

When a magnetic field is applied, the magnetic forces partially cancel out deep in the pile. Motion happens by shearing deep within the pile, in addition to motion close to the surface. We have shown that the slope angle has only a weak dependence on the magnetic field because shear deep in the pile prevents the angle of repose from increasing substantially.

We have investigated the effect of different implementations of friction with the front and back walls of the container. We conclude that the choice of friction model dramatically affects both the zero-field repose angle and its rate of increase with cohesion. Depth-dependent friction causes an increase in the zero-field repose angle and in the gradient $d\theta_r/dR$. Depth-independent friction causes an increase in the zero-field repose angle, but $d\theta_r/dR$ remains unchanged.

We have also demonstrated that in a 3D system, particles near the front and back walls of the container experience a net attractive force pulling them toward the bulk of the pile and away from the walls. We suggest that this attraction will reduce the effect of wall friction on both the repose angle and its rate of increase with cohesion.

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